

NON-REALIZABILITY OF THE POISSON BOUNDARY

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ABSTRACT. We show that for any countable group G equipped with a probability measure μ , there exists a randomized stopping time τ such that (G, μ_τ) admits a strictly larger space of bounded harmonic functions than (G, μ) , unless this space is trivial for all measures on G . In particular, we exhibit an irreducible probability measure on the free group F_2 such that the Poisson boundary is strictly larger than the geometric boundary equipped with the hitting measure, resolving a longstanding open problem. As another consequence, there is never a nontrivial universal topological realization of the Poisson boundary for any countable group.

1. INTRODUCTION

Let G be a countable group equipped with a probability measure μ . We say the measure μ is irreducible if its support generates G as a group. A function $h : G \rightarrow \mathbb{R}$ is μ -harmonic if h satisfies the mean-value property $h(x) = \sum_{g \in G} h(xg)\mu(g)$. The space of bounded μ -harmonic functions can be encoded as the space of bounded measurable functions on a canonical measure space associated to the pair (G, μ) . This space is known as the *Poisson boundary* of (G, μ) , which we denote by $\Pi(G, \mu)$.

Probabilistically, the Poisson boundary captures stochastically significant asymptotic events of the μ -random walk, which is the Markov chain $(w_i)_{i \geq 1}$ on G with transition probabilities $p(g, h) = \mu(g^{-1}h)$. Analytically, the Poisson boundary is a measure G -space with the property that for any bounded μ -harmonic function h on G , there exists a bounded function $f \in L^\infty(\Pi(G, \mu), \nu)$ such that

$$h(g) = \int_{\Pi(G, \mu)} f(\xi) dg_* \nu(\xi).$$

This is known as the *Poisson representation formula*.

Formally, the Poisson boundary is the quotient of the space of μ -random walk sample paths $(G^{\mathbb{N}}, \mathbb{P})$ by the measurable hull of the orbit equivalence relation of the time shift. That is, two paths $(w_i)_{i \geq 1}, (w'_i)_{i \geq 1}$ are equivalent if there exists $k, k' \in \mathbb{N}$ such that $w_{i+k} = w'_{i+k'}$ for all $i \in \mathbb{N}$. We point the reader to [Kai96, Fur02, Zhe22] for more information on Poisson boundaries.

The Poisson boundary was popularized in Furstenberg's 1963 paper [Fur63b], though its history goes back to the work of Blackwell [Bla55], Feller [Fel56], and Doob [Doo59]. It has been a powerful tool to prove rigidity theorems [Fur67] as well as understanding the large-scale geometry of groups [Ers04, BV05, AABV16, EZ20].

Furstenberg observed that in many situations this abstract measure space can be realized by a natural space associated with the group [Fur63b]. This observation immediately leads to the *topological identification problem*: given a pair (G, μ) , can one find a concrete topological

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"boundary" B of the group such that (i) the μ -random walk on G almost surely converges in the induced bordification, and (ii) the space B with the hitting measure ν is the Poisson boundary of (G, μ) . We contrast this with the *identification problem*, which is formulated in the measure category and asks to show that a concrete μ -boundary (a priori just an equivariant quotient of the Poisson boundary) is in fact isomorphic to the Poisson boundary (see [Kai96, Zhe22]).

A prototypical example is the case of the free group F_2 , where for every irreducible probability measure μ , the μ -random walk almost surely converges to the geometric boundary of the tree ∂F_2 . This fact is essentially due to Furstenberg [Fur63a, Fur67, Section 4]. It is then natural to ask under what conditions on μ this boundary is a model for the Poisson boundary.

Over forty years ago, in their 1983 paper, Kaimanovich and Vershik asked whether the Poisson boundary of the free group can be identified with the geometric boundary for *all* measures. When μ is supported on the generating set of F_2 , this was shown to be true by Dynkin-Maljutov [DM61]. Derrienc [Der75] extended their result to the case of μ with finite support. In a breakthrough paper by Kaimanovich [Kai00], he showed that for any μ with finite entropy and finite logarithmic moment, the Poisson boundary of (F_2, μ) is the hyperbolic boundary equipped with the hitting measure. Finally, this was extended to all finite entropy measures in joint work of the authors with Forghani and Tiozzo [CFFT22].

This question has been reiterated numerous times in the past decade [For15, FT19, EK23, Kai24].

Our primary goal is to resolve this question in the negative.

Theorem 1.1. *There exists an irreducible probability measure on F_2 such that $(\partial F_2, \nu)$ is not the Poisson boundary of (F_2, μ) .*

The primary source of difficulty is an utter lack of general techniques in the presence of infinite entropy. The use of entropic methods in this area was pioneered by Kaimanovich and Vershik [KV83] and has been a central motif of the field. Indeed, with a few notable exceptions, some of which we describe below, almost all Poisson boundary identification results have only applied to measures which have finite entropy. Most have used the entropic criterion of Kaimanovich [Kai00]. All non-entropic results instead rely on specific structures of certain probability measures [Bla55, DM61, EK23]. On the other hand, convergence to the boundary holds for all irreducible measures, even with infinite entropy, so it is plausible that the geometric boundary may still be a model for the Poisson boundary. We show that this is not the case. Our method even extends to the free semigroup, where the random walk exhibits no backtracking (see Remark 2.7).

While we give a self-contained proof of Theorem 1.1 in Section 2, it is also a corollary of the following significantly more general result, which shows that for all countable groups (except those for which the Poisson boundary is always trivial), there is no hope to find a universal boundary for all irreducible measures. That is to say, the existence of the probability measure in Theorem 1.1 has little to do with the structure of the free group, but is actually emblematic of a much more general phenomenon.

To state it, we require some definitions.

Definition 1.2. A nonnegative integer-valued random variable τ is a *randomized stopping time* with respect to a filtration $(\mathcal{F}_i)_{i \geq 1}$ if there is a collection of i.i.d. $\text{Uniform}([0, 1])$ random variables $(U_i)_{i \geq 1}$ such the event $\{\tau \leq n\}$ is measurable with respect to the σ -algebra $\sigma(\mathcal{F}_n, U_1, \dots, U_n)$.

Given a probability measure μ , a random variable τ is a *randomized stopping time for the μ -random walk* if the previous definition holds with $\mathcal{F}_n = \sigma(w_1, \dots, w_n)$. We denote by μ_τ the law of the μ -random walk stopped at the random time τ . Stopping times have been a useful tool in the area since its conception, for example in work of Furstenberg [Fur71, Section 4.3] to relate the Poisson boundary of groups and their corecurrent subgroups, and Kaimanovich-Vershik [KV83, Proposition 6.3] to show triviality of the boundary for random walks on lamplighters over $\mathbb{Z}$ and $\mathbb{Z}^2$.

Randomized stopping times in particular have been used by Kaimanovich [Kai92] to show that Furstenberg's discretization of Brownian motion does not change the Poisson boundary [Fur71] (see also [LS84, Section 8]). Forghani and Kaimanovich prove in upcoming work [FK24] (though proofs are included in Forghani's thesis [For15]) that under some mild moment assumptions on a randomized stopping time τ , the pairs (G, μ) and (G, μ_τ) have the same Poisson boundary. They also claim that standard stopping times always preserve the Poisson boundary, though there is a subtle error in the proof (see Remark 2.9). We point out the distinction of randomized stopping times from stopping times, in that they can use randomness external to the μ -random walk, and more discussion of their differences in the context of random walks on groups is available in [For15]. There are many works exploring the relation between asymptotic properties of (G, μ) and (G, μ_τ) under various assumptions on the stopping time τ [Kai83a, HLT14, For15, For17, FMK19], and we point the reader to [FK24, For15] for more detailed history of the subject.

We also require an algebraic property, whose relation with the Poisson boundary has been observed and exploited in numerous previous works [Jaw04, FHTF19, EK23].

Definition 1.3. A non-trivial group G has the *infinite conjugacy class property*, or ICC property, if each of its non-trivial elements has an infinite conjugacy class.

It is a theorem of Jaworski [Jaw04] that if a group does not have an ICC quotient, then the Poisson boundary $\Pi(G, \mu)$ is trivial for every irreducible measure μ (in the finitely generated case, this was previously proven by Dynkin-Maljutov [DM61]). Such a group is called *Choquet-Deny*. It was recently shown that for every group with an ICC quotient, there is a probability measure with a non-trivial Poisson boundary [FHTF19].

A probability measure is said to be non-degenerate if $\text{supp } \mu$ generates G as a semigroup. A measure is symmetric if $\mu(g) = \mu(g^{-1})$ for all $g \in G$.

Theorem 1.4. *Let G be a countable group with an ICC quotient, and μ be any irreducible probability measure on G . Then there exists a randomized stopping time τ for the μ -random walk such that the Poisson boundary of (G, μ_τ) is strictly larger than that of (G, μ) . If μ is symmetric or non-degenerate, μ_τ can be made so as well.*

We remark that a finitely generated group G has an ICC quotient if and only if it is not virtually nilpotent [FF18, DM56, McL56], so Theorem 1.4 applies to all finitely generated groups of superpolynomial growth. In particular, there is a symmetric and non-degenerate measure on the free group whose Poisson boundary is not the geometric boundary. By Jaworski's theorem [Jaw04], our Theorem 1.4 implies that unless the Poisson boundary is always trivial, one can make it strictly larger via transforming μ by a randomized stopping time. In fact, it was explicitly asked in the thesis of Forghani whether randomized stopping times always preserve the Poisson boundary [For15, Question 4.3.1]; we answer this question in the negative.

This is the first non-realizability result for the Poisson boundary. It was previously unknown whether all groups admitted a universal realization of their Poisson boundary. We not only exhibit an example where this is impossible, but show that it is never possible, unless the Poisson boundary is trivial for all measures. In other words, the identification problem has no universal solution for any group, in the sense that for any group which is not Choquet-Deny, and any irreducible measure μ , there is a randomized stopping time τ such that the Poisson boundary $\Pi(G, \mu)$ is a proper quotient of $\Pi(G, \mu_\tau)$.

As one corollary, we show that main result of [FHTF19] can be achieved via the transformation of *any* irreducible measure by a randomized stopping time, although this new measure may have infinite entropy.

Corollary 1.5. *For any countable group G with an ICC quotient equipped with an irreducible probability measure μ , there exists a randomized stopping time τ such that (G, μ_τ) has a non-trivial Poisson boundary.*

Remark 1.6. The authors of [FHTF19] actually construct a measure μ which is fully supported in the sense that $\mu(g) > 0$ for all group elements $g \in G$. Examining our construction of the randomized stopping time τ in section 4.1, we see that for any fully supported measure μ , the measure μ_τ coming from Theorem 1.4 is also fully supported. On the other hand, our construction when applied to the free group will always produce infinite entropy measures, by [CFFT22, Theorem 1.1]. We expect our general construction to always produce infinite entropy measures.

As mentioned earlier, Theorem 1.4 implies the topological identification problem never has a universal solution without additional conditions on the measure. To elaborate, we define a bordification of G to be a space $G \cup B$ where the action of G on itself extends by homeomorphisms to $G \cup B$. We say that a bordification of G is a *candidate boundary* if there exists an irreducible probability measure μ such that the μ -random walk on G almost surely converges in B . We say a measure μ' is *compatible* with B if the μ' -random walk almost surely converges to a unique point in B . This defines a hitting measure ν on B by $\nu(U) := \mathbb{P}(\lim_{n \rightarrow \infty} w_n \in U)$.

Corollary 1.7. *Let G be a countable discrete group with an ICC quotient. For every candidate boundary B , there is a compatible probability measure μ such that B equipped with the hitting measure is not the Poisson boundary of (G, μ) .*

Proof. Since B is a candidate boundary, there exists an irreducible measure μ which is compatible with B . If (B, ν) is not the Poisson boundary of (G, μ) , we are done. If it is, let μ_τ be the measure coming from Theorem 1.4. Since μ_τ is a randomized stopping time transformation of μ , then the μ_τ -random walk almost surely converges and has the same hitting measure (B, ν) . Since there exists a bounded μ_τ -harmonic function that is not μ -harmonic, we deduce that not every bounded μ_τ -harmonic function can be realized by the Poisson integral formula on (B, λ) . $\square$

It has been emphasized that the Poisson boundary is an inherently measure-theoretic object, and that "attempts to treat the Poisson boundary as a topological space... shroud its true nature" [Kai96]. Our theorem underscores this point: the Poisson boundary never admits a topological realization for all measures.

As mentioned previously, results in the field so far have largely relied on entropic methods—in particular, Kaimanovich's entropy criterion for identification of the Poisson boundary

[Kai00] (the first proof of which appeared in [Kai85]). These most notably include (i) groups with hyperbolic properties [Led85, Kai00, MT18, Hor16, CFFT22] as well as (ii) wreath products and their variants [Ers11, LP21, EF25, EFR23, Sil24, VKS25] (see [Zhe22, FS24] and references therein for more examples).

There are however, a few notable exceptions. First is the original proof of Kaimanovich and Vershik that amenable groups admit irreducible measures with trivial boundary [KV83, Theorem 4.4]. Second is the work of Andrei Alpeev [Alp25], who showed that if a group is not C^* -simple, then the action on the Poisson boundary is not essentially free for a generic measure on the group. In addition there is the work of Alpeev [Alp24a] and Kaimanovich [Kai83b] who showed for infinite entropy measures that the left random walk can have a trivial Poisson boundary even if the right one is nontrivial, and that the Poisson boundary need not split over products in the presence of infinite entropy [Alp24b, Kai24]. Finally, we wish to highlight the recent paper of Erschler and Kaimanovich who, on any ICC group, construct a family of forests and associated probability measures such that the corresponding random walks in some sense resemble the simple random walk on a tree [EK23]. As a result, the boundary of these forests provide models of the Poisson boundary for those particular measures.

We leverage Erschler and Kaimanovich's framework in the proof of Theorem 1.4. They remark that the boundaries they construct do not have any common underlying space, and "this absence of universality seems to be the first example of this kind". In the case of finite entropy measures on the free group, these boundaries coincide up to a conull set with the geometric boundary (see [CFFT22]).

We prove that in infinite entropy, there are infinitely many non-isomorphic Poisson boundaries on the same group. To be precise, we use Erschler and Kaimanovich's framework to construct a boundary for the transformed measure μ_τ , and show that this new boundary is strictly larger than the Poisson boundary of μ . We also show that this boundary is the Poisson boundary of μ_τ (see Corollary 4.8).

We now explain the key observation behind our construction. Let μ be the measure driving the lazy simple random walk on F_2 . If τ is a randomized stopping time for the μ -random walk, then the μ -random walk and the μ_τ -random walk induce the same hitting measure ν on the boundary. If $(\partial F_2, \nu)$ is the Poisson boundary of (F_2, μ_τ) , then the Poisson integral formula implies both μ and μ_τ admit exactly the same collection of bounded harmonic functions. Hence to show that $(\partial F_2, \nu)$ is not the Poisson boundary, it suffices to exhibit a bounded function that is μ_τ -harmonic but not μ -harmonic. In other words, we completely bypass the need to work with the conditional random walk or with any entropic criteria.

The paper is organized as follows. In Section 2 we provide a self-contained proof of Theorem 1.1. In Section 3 we prove a lemma on the abundance of "switching" elements, akin to [FHTF19], [EK23], [GMBT24], and [LBMB18]. In Section 4 we recall the framework of Erschler and Kaimanovich and use our lemma to construct the measure μ_τ . Finally in Section 5 we prove Theorem 1.4 by exhibiting an explicit μ_τ -harmonic function that is not μ -harmonic.

2. PROOF OF THEOREM 1.1

We let μ be the measure driving the lazy simple random walk on F_2 , that is $\mu = \frac{1}{2}\delta_e + \frac{1}{8}\sum_{s \in \{a^\pm, b^\pm\}} \delta_s$. We will construct a randomized stopping time τ for the μ -random walk.

Our randomized stopping time τ will be a mixture of a countable collection of stopping times $(\tau_i)_{i \geq 0}$, weighted by an auxiliary probability measure p on the natural numbers $\mathbb{N}$. That is to say, one step of the μ_τ random walk can be sampled by first sampling some $i \sim p$, then sampling a group element according to the stopped measure μ_{τ_i} . In particular, we can write $\mu_\tau = \sum_{i \geq 0} p(i) \mu_{\tau_i}$. For the remainder of this section, we will specify what properties we want τ to satisfy, and then we will construct τ via a judicious choice of p and a sequence of stopping times $(\tau_i)_{i \geq 0}$.

2.1. A reduction to the lamplighter group. Recall the lamplighter group $\mathbb{Z} \wr \mathbb{Z}/2\mathbb{Z}$, defined as the semidirect product $\mathbb{Z} \ltimes \bigoplus_{i \in \mathbb{Z}} \mathbb{Z}/2\mathbb{Z}$, where the action of $\mathbb{Z}$ is by shifting the index. Elements of this group are pairs (X, φ) where φ is a finitely supported function $\varphi : \mathbb{Z} \rightarrow \mathbb{Z}/2\mathbb{Z}$ and X is an integer.

We consider the homomorphism π from F_2 to the lamplighter $\mathbb{Z} \wr \mathbb{Z}/2\mathbb{Z}$ induced by setting $\pi(a) = (e_0, 0)$ and $\pi(b) = (0, 1)$, where $e_0(x) = 1$ if $x = 0$ and $e_0(x) = 0$ otherwise.

Our choice of p and $(\tau_i)_{i \geq 0}$ will be to ensure that the following proposition holds:

Proposition 2.1. *The Poisson boundary of $(\mathbb{Z} \wr \mathbb{Z}/2\mathbb{Z}, \pi_* \mu_\tau)$ is nontrivial.*

We explain why this implies that the geometric boundary of F_2 is not a model for the Poisson boundary of (F_2, μ_τ) .

Proof of Theorem 1.1 given Proposition 2.1. Let ν be the hitting measure on ∂F_2 induced by the lazy simple random walk. As μ_τ is a randomized stopping time transformation of the lazy simple random walk, the measure μ_τ induces the same hitting measure on ∂F_2 . If $(\partial F_2, \nu)$ is the Poisson boundary of (F_2, μ_τ) , then the Poisson integral formula implies that μ and μ_τ admit the same collection of bounded harmonic functions.

Hence to show the claim it suffices to exhibit a bounded μ_τ -harmonic function that is not μ -harmonic. Since the Poisson boundary of $(\mathbb{Z} \wr \mathbb{Z}/2\mathbb{Z}, \pi_* \mu_\tau)$ is nontrivial, we may let f be a non-constant bounded $\pi_* \mu_\tau$ -harmonic function on $\mathbb{Z} \wr \mathbb{Z}/2\mathbb{Z}$. Then the pullback $\pi^* f$ is non-constant, bounded and μ_τ -harmonic.

We claim $\pi^* f$ is not μ -harmonic. To deduce this, we show that any bounded function on F_2 which is μ -harmonic and constant on the fibers of π must be constant. Indeed, we can write such a function in the form $\pi^* g$ where g is a bounded function on $\mathbb{Z} \wr \mathbb{Z}/2\mathbb{Z}$. Then for any $x \in F_2$, we have

$$\sum_{y \in \mathbb{Z} \wr \mathbb{Z}/2\mathbb{Z}} g(\pi(x)y) \pi_* \mu(y) = \sum_{\substack{y \in \mathbb{Z} \wr \mathbb{Z}/2\mathbb{Z}, \\ k \in \pi^{-1}(y)}} \pi^* g(xk) \mu(k) = \pi^* g(x) = g(\pi(x)).$$

Hence g is $\pi_* \mu$ -harmonic. Since $\pi_* \mu$ is a finitely supported symmetric and irreducible measure on the lamplighter, the Poisson boundary of $(\mathbb{Z} \wr \mathbb{Z}/2\mathbb{Z}, \pi_* \mu)$ is trivial. As a result, g is constant. As $\pi^* g$ is constant on the fibers of π , it must be constant everywhere. $\square$

Remark 2.2. As pointed out to us by Vadim Kaimanovich, Proposition 2.1 implies Theorem 1.1 by the fact that for any normal subgroup N of a countable group G , the Poisson boundary $\Pi(G/N, \pi_* \mu)$ is the space of ergodic components of the action of N on $\Pi(G, \mu)$ (see [Kai81] or [For15, Theorem 2.4.2]). Hence if the Poisson boundary of $(\mathbb{Z} \wr \mathbb{Z}/2\mathbb{Z}, \pi_* \mu_\tau)$ is larger than that of $(\mathbb{Z} \wr \mathbb{Z}/2\mathbb{Z}, \pi_* \mu)$, then the Poisson boundary of (F_2, μ_τ) is strictly larger than the geometric boundary. For the convenience of the reader, we decided to retain the proof above.

We now proceed with the construction of p and $(\tau_i)_{i \geq 0}$.

2.2. A heavy-tailed probability measure on $\mathbb{N}$. Given a probability measure p on $\mathbb{N}$, let $(X_i)_{i \geq 1}$ be i.i.d samples drawn according to p . Let T_k be the k th record time given by $T_0 = 1, T_k = \inf\{i > T_{k-1}, X_i \geq X_j \forall j < i\}$. In addition, let $R_k = X_{T_k}$ be the k th record value.

Definition 2.3. A probability measure p on $\mathbb{N}$ has *eventually simple records* if almost surely, for k sufficiently large, we have $R_{k+1} > R_k$.

We record some lemmas about records times of i.i.d. samples.

Lemma 2.4. [Ver73, BSW94, Qi97, Eis09, EK23, Lemma 2.3] *There exists a probability measure p on $\mathbb{N}$ which satisfies $p(i) > 0$ for all $i \geq 0$ and has eventually simple records.*

Remark 2.5. In fact, there are many probability measures with eventually simple records. It was observed by Vervaat (see [Ver73]) that the sequence of record values $(R_k)_{k \geq 0}$ is a Markov chain with transition probabilities $p(i, j) := p(j) / \sum_{\ell \geq i} p(\ell)$. By Borel-Cantelli, a sufficient condition for eventual record simplicity is that $\sum_{i \geq 0} p(i, i)^2 < \infty$ and that p has infinite support. This is satisfied, for instance, by the measure $p(n) = c(n+1)^{-2}$ for an appropriate constant $c > 0$.

For the rest of this section, fix a measure p satisfying the conclusion of Lemma 2.4.

We also introduce a gauge function which controls the record times in terms of previous record values.

Lemma 2.6. [EK23, Lemma 2.17] *For any probability measure p on $\mathbb{N}$, there exists a non-decreasing $\Phi : \mathbb{N} \rightarrow \mathbb{N}$ such that almost surely for k sufficiently large we have $T_{k+1} < \Phi(R_k)$.*

We note that the condition here is on T_{k+1} , not T_k .

2.3. A sequence of stopping times. Let Z_n denote the simple random walk on F_2 and $(X, \varphi_n) = \pi(Z_n)$ its projection to $\mathbb{Z} \wr \mathbb{Z}/2\mathbb{Z}$. Given sequences of natural numbers $(s_n)_{n \geq 0}, (r_n)_{n \geq 0}$, we define a sequence of stopping times $(\tau_n)_{n \geq 0}$ by

$$\tau_n = \inf \{t \geq 1, \varphi_t|_{[-s_n, s_n]} \equiv 0, |X_t| \geq r_n\}.$$

In other words, the first time that all the lamps in the interval $[-s_n, s_n]$ are off and the lamplighter is more than r_n away from the origin.

Since the simple random walk on $\mathbb{Z}$ is recurrent, it follows that each τ_n is almost surely finite. Indeed, recurrence of the simple random walk implies the recurrence of the pair $(\varphi_t|_{[-s_n, s_n]}, X_t)$, so that almost surely the condition in the definition of τ_n is met infinitely often.

We proceed to inductively make a choice of $s_1 < r_1 < s_2 < r_2 < \dots$. Set $s_1 = r_1 = 0$. Let $r_1, s_1, \dots, s_{k-1}, r_{k-1}$ be given. Given a sequence $(i_1, i_2, \dots, i_{\Phi(k)}) \subset \{1, \dots, k-1\}^{\Phi(k)}$, consider the random sequence $(g_j)_{1 \leq j \leq \Phi(k)}$ where the g_j are each drawn independently from $\mu_{\tau(j)}$, the transformation of μ by the stopping time $\tau^{(j)} = \tau_{i_j}$, which is well defined as each element of the sequence is less than k and we have already chosen $r_1, s_1, \dots, r_{k-1}, s_{k-1}$. Then for $1 \leq j \leq \Phi(k)$, let $w_j = g_1 \dots g_j$, and let φ_j be the lamp configuration of $\pi(w_j)$.

We pick s_k large enough that for every choice of sequence $(i_1, \dots, i_{\Phi(k)})$, with probability at least $1 - 2^{-k}$ the support of φ_j is contained in $[-s_k/3, s_k/3]$ for all $1 \leq j \leq \Phi(k)$. Then we set $r_k = 3s_k$.

With our sequence of stopping times and probability measure p chosen, we are now ready to prove proposition 2.1 and hence Theorem 1.1.

Proof of Proposition 2.1. First observe that μ_τ is irreducible as $p(0) > 0$ and μ is irreducible. We show the Poisson boundary of $(\mathbb{Z} \wr \mathbb{Z}/2\mathbb{Z}, \pi_*\mu_\tau)$ is nontrivial by showing that the sequence of lamp configurations $(\varphi_j)_{j \geq 0}$ almost surely converges. Let $(X_i)_{i \geq 1}$ be i.i.d samples from p , and $(T_k)_{k \geq 1}$ the associated record times. Then almost surely there exists k_0 such that for $k \geq k_0$, we have $R_{k-1} < R_k < R_{k+1}$, and $T_{k+1} < \Phi(R_k)$ (by Lemma 2.6). Given some $k \geq k_0$, consider the random walk at time $T_{k+1} - 1$,

$$w_{T_{k+1}-1} = \underbrace{g_1 \cdots g_{T_k-1}}_a g_{T_k} \underbrace{g_{T_k+1} \cdots g_{T_{k+1}-1}}_b.$$

Observe that each of a and b is a product of at most $\Phi(R_k)$ increments, drawn independently from some sequence $\mu_{\tau_{i_1}}, \dots, \mu_{\tau_{i_\ell}}$ where $i_j < R_k$ for all $1 \leq j \leq \ell$. Then by our choice of $(s_k)_{k \geq 0}, (r_k)_{k \geq 0}$ together with the Borel-Cantelli lemma, we know that almost surely for k sufficiently large, we have

$$\varphi_{T_k-1}|_{[-s_{R_k}/3, s_{R_k}/3]} = \varphi_j|_{[-s_{R_k}/3, s_{R_k}/3]}$$

for all $T_k \leq j \leq T_{k+1} - 1$. Since the records R_k are by definition increasing and they almost surely go to infinity, the limit $\lim_{j \rightarrow \infty} \varphi_j$ exists almost surely.

As μ_τ is irreducible, the limiting lamp configuration is not almost surely constant, so the Poisson boundary of $(\mathbb{Z} \wr \mathbb{Z}/2\mathbb{Z}, \pi_*\mu)$ is nontrivial. $\square$

Remark 2.7. We remark that this construction can also be made to work for the free semigroup. For the semigroup F_2^+ generated by a and b , we get a surjection onto the lamplighter by sending a to $(0, 1)$ and b to $(e_0, -1)$. The simple random walk on the free semigroup pushes forward to a irreducible random walk on the lamplighter whose projection to $\mathbb{Z}$ is recurrent, so a similar choice of randomized stopping time works to produce a measure on F_2^+ with Poisson boundary strictly larger than the geometric boundary.

Remark 2.8. We can make our construction work with a non-randomized stopping time as opposed to a randomized stopping time. To do this, we exploit the randomness coming from the number of increments of the μ -random walk which take the value e . Indeed, observe that the supports of $(\mu_{\tau_i})_{i \geq 0}$ are pairwise disjoint, so that we may let τ be the stopping time given by stopping the μ -random walk the first time it lies in the support of some μ_{τ_n} if the number of identity increments is at most $\lfloor 1/p(n) \rfloor$ modulo $\lfloor 1/p(n) \rfloor^2$. If $(s_k)_{k \geq 1}, (r_k)_{k \geq 1}$ are sufficiently large depending on p , then the number of identity increments modulo $\lfloor 1/p(n) \rfloor^2$ is asymptotically uniformly distributed and asymptotically independent of the position of the random walk, so that our same proof works to show that the induced random walk on the lamplighter has non-trivial boundary.

Remark 2.9. It is claimed in an unpublished manuscript of Forghani and Kaimanovich [FK24], and quoted in [For15, Theorem 3.6.1], that non-randomized stopping times do not modify the Poisson boundary. However there is a subtle error in the proof. The transformed random walk on the group G is lifted to the free semigroup on the set $\text{supp } \mu$, viewed as abstract symbols, where the Poisson boundary is left unchanged by a lift of the stopping time. The transformed measure on the free semigroup, however, is no longer generating, so one can not necessarily conclude that the Poisson boundary of the quotient (G, μ_τ) is left unchanged.

3. RANDOM WALKS AND SWITCHING ELEMENTS IN ICC GROUPS

Given a finite subset $F \subset G$ and a (not necessarily finite) subset $A \subset G$, we say that A is *F-switching* if for any $f_1, f_2, f_3, f_4 \in F, a_1, a_2 \in A$, the equality $f_1 a_1 f_2 = f_3 a_2 f_4$ implies $f_1 = f_3, f_2 = f_4$, and $a_1 = a_2$. We say that an element $g \in G$ is *F-switching* if the set $\{g\}$ is *F-switching* and that g is *F-superswitching* if the set $\{g, g^{-1}\}$ is *F-switching*. Switching sets were defined in [EK23], generalizing the notion of switching elements from [FHTF19, FTVF19],

The goal of this section is to prove Lemma 3.3 where we produce, for any finite $F \subset G$, a stopping time τ such that the support of μ_τ is *F-switching*. This lemma is key to the construction of our randomized stopping time in the proof of Theorem 1.4, and is applied in Section 4.1.

Without loss of generality we may assume that $\mu(e) > 0$, since replacing μ with $\frac{1}{2}\mu + \frac{1}{2}\delta_e$ preserves the space of bounded harmonic functions. One may interpret this as first transforming μ by a randomized stopping time which is 0 or 1 each with probability 1/2, then transforming it by the randomized stopping time τ .

The authors of [FHTF19] and [EK23] construct switching and superswitching elements using a density argument, that if a group is ICC then the set of non-*F-superswitching* elements is small in an appropriate sense, and so there are infinitely many *F-superswitching* elements. However, to construct a measure using a randomized stopping time, it is not enough to show the existence of switching elements. Instead we need to show that the μ -random walk witnesses an abundance of switching elements. To do this, we take an approach inspired by [GMBT24]. Roughly speaking, we show that any irreducible random walk run for sufficiently long will be *F-switching* with high probability. We need the following lemma, similar to [GMBT24, Claim 2.1] and [LBMB18, Lemma 3.2].

Lemma 3.1. *For any irreducible probability measure μ on a countable group G , any subgroup $H < G$ of infinite index and finite subset $F \subset G$, we have $\mathbb{P}(w_n \in HF) \rightarrow 0$.*

Proof. Consider the action of G on the space of right cosets G/H . We first claim that every $\text{supp } \mu$ -semigroup orbit is infinite. Indeed, if there was a finite set X such that $Xg \subset X$ for all $g \in \text{supp } \mu$, then we would have $Xg = X$ and so $X = Xg^{-1}$. Therefore the group generated by $\text{supp } \mu$ would fix X , which would imply H has finite index.

Now consider the (directed) transition graph of the Markov chain driven by the μ -random walk on the space of right cosets. For the claim to fail, there must exist some Hf contained in some strongly irreducible component $\mathcal{S}$. Since the $\text{supp } \mu$ -semigroup orbit is infinite, the counting measure on $\mathcal{S}$ is an infinite stationary measure, so the ergodic theorem for aperiodic and irreducible Markov chains implies $\mathbb{P}(w_n \in Hf) \rightarrow 0$. The Markov chain is aperiodic as we assumed $\mu(e) > 0$.

Applying the union bound over the finitely many $f \in F$ gives the claim. $\square$

Using this lemma, we show that random walks are *F-switching* with high probability.

Lemma 3.2. *For any irreducible probability measure μ on a countable ICC group G and finite subset $F \subset G$, we have*

$$\mathbb{P}(w_n \text{ is } F\text{-switching}) \rightarrow 1.$$

If μ is non-degenerate, then

$$\mathbb{P}(w_n \text{ is } F\text{-superswitching}) \rightarrow 1.$$

Proof. Without loss of generality we may assume F is symmetric, as any element that is $(F \cup F^{-1})$ -switching must be F -switching. For an element a to not be F -switching, we must have $f_1 a f_2 = f_3 a f_4$ for some $(f_1, f_2) \neq (f_3, f_4) \in F \times F$, hence $a f_2 f_4^{-1} a^{-1} = f_1^{-1} f_3$. Given any non-trivial element $f \in F^2$, the collection of $a \in G$ such that $a f a^{-1} \in F^2$ is contained in a union of at most $|F|^2$ cosets of the centralizer $C_G(f)$. As G has the ICC property, the subgroup $C_G(f)$ has infinite index, so Lemma 3.1 implies $\mathbb{P}(w_n f w_n^{-1} \in F^2) \rightarrow 0$. Applying the union bound over $f \in F^2 \setminus \{e\}$ we see that $\mathbb{P}(w_n \text{ is } F\text{-switching})$ tends to 1.

Now suppose that μ is non-degenerate. For $x, y \in F$, let $S_{x \rightarrow y} = \{g \in G, g x g = y\}$, so by the union bound we are done if we show that $\mathbb{P}(w_n \in S_{x \rightarrow y}) \rightarrow 0$ for all $(x, y) \neq (e, e)$.

Suppose for the sake of contradiction that $\limsup \mathbb{P}(w_n \in S_{x,y}) > 0$ for some $(x, y) \neq (e, e)$. Let $g \in S_{x,y}$, then any element h contained in $g^{-1} S_{x,y}$ must be in the centralizer of $(xg)^2$ (see [FHTF19, Proposition 2.5]). Since $\limsup \mathbb{P}(w_n \in S_{x,y}) > 0$, then as g is in the semigroup generated by $\text{supp } \mu$ we deduce that $\limsup \mathbb{P}(w_n \in g^{-1} S_{x,y}) > 0$ as well. Hence $\limsup \mathbb{P}(w_n \in C_G((xg)^2)) > 0$ so we see from Lemma 3.1 that $C_G((xg)^2)$ must have finite index. Since G is ICC, this implies $(xg)^2 = e$. Letting I be the set of involutions, we see that $x S_{x,y} \subset I$ so in particular $\limsup \mathbb{P}(w_n \in I) > 0$. However, since G is ICC, it is in particular not virtually abelian, so by Theorem 5.1 of [ABGK23], we have $\mathbb{P}(w_n \in I) \rightarrow 0$, a contradiction. $\square$

It would be tempting to construct μ_τ by letting τ be the first time that the μ -random walk is F -switching. This would produce a measure μ_τ such that every element of $\text{supp } \mu_\tau$ is F -switching. However, we need to produce a measure μ_τ such that the set $\text{supp } \mu_\tau$ is F -switching, which is a strictly stronger property. We construct such a measure in the following lemma.

Lemma 3.3. *Let G be a countable ICC group. For any finite set $F \subset G$ and irreducible probability measure μ on G , there exists an almost surely finite (non-randomized) stopping time τ with the property that $\text{supp } \mu_\tau$ is F -switching. Moreover if μ is symmetric, then μ_τ can be made symmetric.*

Proof. Without loss of generality we may assume F is symmetric, as any set which is $(F \cup F^{-1})$ -switching is F -switching. Suppose μ is irreducible (resp. irreducible and symmetric). We first ‘thin’ out the set of all F -switching (resp. superswitching) elements as follows. Let $(U_x)_{x \in G}$ be a collection of i.i.d. $\text{Uniform}([0, 1])$ random variables, and let T be the subset of G consisting of all a that are F -switching (resp. F -superswitching) and satisfy $U_a \geq U_b$ for all $b \in F^2 a F^2$ (resp. $F^2 a^\pm F^2$). Almost surely the random variables are distinct, so the set T (resp. $T \cup T^{-1}$) is F -switching. We let τ be the first hitting time of T (resp. $T \cup T^{-1}$), so it suffices to show that τ is almost surely finite.

By Lemma 3.2, almost surely for every sample path of the μ -random walk, w_n is F -switching infinitely often. If μ is irreducible and symmetric, it is non-degenerate, so w_n is F -superswitching infinitely often. Observe that the collection of subsets $\{F^2 a F^2, a \text{ is } F\text{-switching}\}$ (resp. $\{F^2 a^\pm F^2, a \text{ is } F\text{-superswitching}\}$) induces a partition of the set of F -switching (resp. F -superswitching) elements into sets of size at most $2|F|^4$. Then the sample path will almost surely intersect infinitely many distinct such subsets $A_1, A_2, \dots$ in elements $a_1, a_2, \dots$. Since the random variables $(U_x)_{x \in G}$ are independent of the random walk, then each a_i has probability at least $|F|^{-4}/2$ of being in the random set T , independent of all other a_j . Letting τ be the first hitting time of T (resp. $T \cup T^{-1}$), we see that τ is almost surely finite.

We observe that if μ is symmetric then, as $T \cup T^{-1}$ is symmetric, the stopped measure μ_τ is clearly symmetric as well. By construction the set $\text{supp } \mu_\tau$ is F -switching. $\square$

4. ARBOREAL STRUCTURES ON GROUPS

In this Section we construct our randomized stopping time used in the proof of Theorem 1.4. Using the framework developed in [EK23], we produce a μ_τ -boundary which, in section 5, enables us to construct a μ_τ -harmonic function that is not μ -harmonic. A consequence of the framework of [EK23] is a short proof that this μ_τ -boundary is the Poisson boundary of (G, μ_τ) (see Corollary 4.8).

We first recall some notions from [EK23].

Definition 4.1. A *scale* on countable group is a triple $\Lambda = (\lambda, \Sigma, \mathbf{A})$ where $\lambda : \mathbb{N} \rightarrow \mathbb{N}$ is a non-decreasing function, and $\Sigma = (\Sigma_1, \Sigma_2, \dots)$ and $\mathbf{A} = (A_0, A_1, \dots)$ are two sequences of subsets, where $(\Sigma_n)_{n \geq 1}$ are pairwise disjoint and A_0 contains the identity.

Given a scale, we define

$$\Delta_n = A_0 \cup \bigcup_{i < n} (\Sigma_i^\pm \cup A_i^\pm).$$

Observe that in the definition of Δ_n , the union is taken over i strictly less than n , and that Δ_n contains the identity.

Definition 4.2. A *spike decomposition* of a group element $g \in G$ with respect to a scale $(\lambda, \Sigma, \mathbf{A})$ is a triple $(\overleftarrow{g}, \widehat{g}, \overrightarrow{g})$ with $g = \overleftarrow{g} \widehat{g} \overrightarrow{g}$, where $\widehat{g} \in \Sigma_n$ for some n , and the prefix and postfix $\overleftarrow{g}, \overrightarrow{g}$ are products of at most $\lambda(n)$ elements from Δ_n .

Definition 4.3. The *despiking graph* of a scale Λ is the graph with vertex set G and edges $(\overleftarrow{g}, g)$ where $\overleftarrow{g}$ is the prefix of a spike decomposition of an element $g \in G$.

The following definition is slightly stronger than that in [EK23], but suffices for our purposes.

Definition 4.4. We say a scale $\Lambda = (\lambda, \Sigma, \mathbf{A})$ is a *ladder* if

- (1) Any $g \in G$ admits at most one spike decomposition.
- (2) For all $n \geq 1$, the sets Σ_n and $\Delta_n^{3\lambda(n)}$ are disjoint.

Proposition 4.5. For any ladder Λ on a group G ,

- (1) The despiking graph $\mathcal{F}$ is a forest.
- (2) Any tree of the forest $\mathcal{F}$ contains a unique unspiked vertex

We root each tree in $\mathcal{F}$ at its unique unspiked vertex, and refer to $\mathcal{F}$ as the *ladder forest* associated to Λ .

The following lemma is the raison d'être of arboreal structures.

Lemma 4.6. [EK23, Lemma 4.5] If $\Lambda = (\lambda, \Sigma, \mathbf{A})$ is a scale on a countable group G such that for all $n \geq 1$, Σ_n is $\Delta_n^{5\lambda(n)}$ -switching, then Λ is a ladder.

4.1. Construction of μ_τ . Let G be an ICC group equipped with an irreducible probability measure μ .

By Lemma 2.4, we fix a probability measure p on the natural numbers $\mathbb{N}$ such that $p(i) > 0$ for all $i \in \mathbb{N}$ and records are eventually simple. We also recall our gauge function Φ from Lemma 2.6.

We inductively define a sequence of stopping times τ_i for $i \geq 0$. First we let $\tau_0 = 1$, and given $\tau_0, \dots, \tau_{k-1}$ we define τ_k as follows. We inductively keep track of a collection of finite subsets $(S_{i,j})_{i,j \geq 1}$ of G such that $\mu_{\tau_i}(S_{i,j}) > 1 - \Phi(j)^{-1}2^{-j}p_i$ and $S_{i,j-1} \subseteq S_{i,j}$. We define this collection by choosing $(S_{i,k})_{i \geq 1}$ after choosing τ_k . Also let $S_{0,j} = \{e\}$.

We let τ_k be the stopping time coming from Lemma 3.3 taking input the finite set

$$\left(\bigcup_{0 \leq i \leq k-1} S_{i,k} \right)^{5\Phi(k)}.$$

Denote by A_i the set inside the parentheses in the previous expression. Letting $\Sigma_i := S_{i,i}$, we see from Lemma 4.6 that the scale $(\Phi, \Sigma, \mathbf{A})$ is a ladder.

We define a randomized stopping time τ as a mixture of the stopping times $(\tau_i)_{i \geq 0}$ weighted by p . In other words, we have

$$\mu_\tau = \sum_{i \geq 0} p(i) \mu_{\tau_i}.$$

4.2. Convergence to the boundary. Since the scale $(\Phi, \Sigma, \mathbf{A})$ is a ladder, by Lemma 4.6 we can consider the associated ladder forest $\mathcal{F}$. We show that the μ_τ -random walk almost surely converges to the boundary of this forest. This is a very slight modification of the proof of [EK23, Theorem 3.1], included mainly for the convenience of the reader.

Proposition 4.7. *For any $g \in G$, the μ_τ -random walk started at g almost surely converges to the boundary of the forest $\partial\mathcal{F}$. In addition, almost every sample path visits all but finitely many points in the corresponding geodesic ray in $\mathcal{F}$.*

Proof. We sample the μ_τ -random walk in the following way. First we draw $(X_i)_{i \geq 1}$ i.i.d. according to p . Then, conditional on the X_i 's, we sample $g_i \sim \mu_{\tau_{X_i}}$. By Lemmas 2.4 and 2.6, there almost surely exists k_0 such that for all $k \geq k_0$, the k th record time satisfies $T_{k+1} \leq \Phi(R_k)$ and the record R_k is simple.

Let E_k be the event that for every $1 \leq i \leq \Phi(k)$, the increment g_i lies in $S_{X_i,k}$. Then we have $\mathbb{P}(E_k^c) \leq 2^{-k}$. In addition, g_{T_k} lies in Σ_{R_k} with probability at least $1 - p_{R_k} 2^{-R_k}$. By Borel-Cantelli, almost surely for all k sufficiently large the event E_k holds and $x_{T_k} \in \Sigma_{R_k}$. Increasing k_0 if necessary, suppose this holds for all $k \geq k_0$.

Examine the random walk at time n satisfying $T_k \leq n \leq T_{k+1} - 1$.

$$w_n = \underbrace{g_1 \dots g_{T_k-1}}_a g_{T_k} \underbrace{g_{T_k+1} \dots g_n}_b.$$

Then each of a and b is a product of at most $\Phi(R_k)$ elements from Δ_{R_k} , hence $ag_{T_k}b$ is a spike decomposition of w_n with spike g_{T_k} and prefix $a = w_{T_k-1}$. As this holds for all $k \geq k_0$, the sequence $(w_{T_k-1})_{k \geq k_0}$ is a geodesic ray in $\mathcal{F}$, w_n converges to the boundary of $\mathcal{F}$ and w_n intersects all but finitely many points of $(w_{T_k-1})_{k \geq k_0}$.

Now suppose we start the random walk at $g \in G$. Since $g \in \Delta_\ell$ for ℓ sufficiently large, and almost surely infinitely many of the increments g_i are equal to the identity as $\mu(e), p(0) > 0$,

the previous argument shows that increment $gw_n = gag_{T_k}b$ is still spiked with prefix ga and spike g_{T_k} for k sufficiently large. $\square$

Using the “trunk criterion” of Erschler and Kaimanovich, we can show the following

Corollary 4.8. *Let ν be the hitting measure on the boundary of the forest $\mathcal{F}$. Then $(\partial\mathcal{F}, \nu)$ is a model for the Poisson boundary of (G, μ_τ) .*

Proof. By Proposition 4.7, for ν -a.e. $\xi \in \partial\mathcal{F}$, the conditional random walk almost surely hits all but finitely many points on the unique geodesic in $\mathcal{F}$ converging to ξ . In the language of [EK23], this implies the geodesic forms a trunk for the conditional random walk, so by [EK23, Proposition 3.8], the space $(\partial\mathcal{F}, \nu)$ is the Poisson boundary of (G, μ_τ) . $\square$

5. PROOF OF THEOREM 1.4

Let G be a countable group with an ICC quotient, equipped with an irreducible probability measure μ . Without loss of generality, we may assume that G itself has the ICC property. Indeed, any group with an ICC quotient carries a canonical normal subgroup H , known as the hyper-FC-Centre, with the property that G/H is ICC [FF18, Proposition 2.2] and that for any irreducible measure μ , the Poisson boundaries of (G, μ) and $(G/H, \pi_*\mu)$ coincide (see [EK23, Proposition 5.1], [Jaw04, Lemma 4.7]), where $\pi : G \rightarrow G/H$ is the quotient map. Given a randomized stopping time τ for the $\pi_*\mu$ -random walk on the quotient group, we may pull back to G to get a randomized stopping time for the μ -random walk.

To deduce Theorem 1.4, it suffices to exhibit a bounded μ_τ -harmonic function on G that is not μ -harmonic. Recall that we can sample the i th step of our μ_τ random walk by choosing $X_i \sim p$ and then choosing $w_i \sim \mu_{\tau_{X_i}}$.

Given a natural number $n \in \mathbb{N}$, we can construct a μ_τ -harmonic function as follows: given the limiting boundary point $\xi \in \partial\mathcal{F}$ we ask what is the conditional probability that the first step of the μ_τ -random walk was drawn from the measure μ_{τ_n} . This produces a bounded function on $\partial\mathcal{F}$, which induces a μ_τ -harmonic function on G by integrating against the μ_τ -stationary hitting measure ν . Explicitly, this produces the μ_τ -harmonic function $f_n(g) = \int_{\partial\mathcal{F}} \mathbb{P}^\xi(X_1 = n) dg_*\nu(\xi)$. Here, $(\mathbb{P}^\xi)_{\xi \in \partial\mathcal{F}}$ are the measures coming from the disintegration $\mathbb{P} = \int_{\partial\mathcal{F}} \mathbb{P}^\xi d\nu(\xi)$ [Roh67, Kai00]. However, we find it convenient to use a slightly different μ_τ -harmonic function.

If the supports of $(\mu_{\tau_i})_{i \geq 0}$ were pairwise disjoint, then we could express these conditional probabilities using the Doob h -transform to see that

$$\begin{aligned} \mathbb{P}^\xi(X_1 = n) &= \sum_{y \in \text{supp } \mu_{\tau_n}} \mathbb{P}^\xi(w_1 = y) \\ &= \sum_{y \in G} p(n) \frac{dy_*\nu}{d\nu}(\xi) \mu_{\tau_n}(y) \end{aligned}$$

The supports of $(\mu_{\tau_i})_{i \geq 0}$ need not be disjoint (say, if μ_0 is fully supported), though the latter expression still allows us to define a bounded μ_τ -harmonic function. Explicitly, our choice of bounded μ_τ -harmonic function is

$$h_n(g) := p(n) \cdot \int_{\partial\mathcal{F}} \sum_{y \in G} \frac{dy_*\nu}{d\nu}(\xi) \mu_{\tau_n}(y) dg_*\nu(\xi).$$

Observe that $h_n(e) = p(n)$ as ν is μ_τ -stationary. If h_n was also μ -harmonic, then the Optional Stopping Theorem would imply that the average $\sum_{g \in G} \mu_{\tau_n}(g) h_n(g)$ is also equal to $p(n)$. We will show that this average is at least $1 - o_n(1)$, which is strictly greater than $p(n)$ for n sufficiently large. This will be a consequence of the following proposition, proven in Section 5.1. Recall the sets $S_{n,n}$ defined in Section 4.1.

Proposition 5.1. *Let $\mathcal{F}_n$ be the collection of subtrees consisting of descendants of elements of the form gw where $g \in S_{n,n}$ and $w \in \Delta_n^{\Phi(n)}$. Then for any $g \in S_{n,n}$, we have $g_*\nu(\partial\mathcal{F}_n) \geq 1 - o_n(1)$.*

Given this proposition, we perform a little bit of averaging. Expanding the definition of $h_n(g)$, swapping a sum and integral, and changing variables gives us

$$\sum_{g \in G} \mu_{\tau_n}(g) h_n(g) = p(n) \cdot \int_{\partial\mathcal{F}} \left(\sum_{g' \in G} \mu_{\tau_n}(g') \frac{dg'_*\nu}{d\nu}(\xi) \right) \left(\sum_{g \in G} \mu_{\tau_n}(g) \frac{dg_*\nu}{d\nu}(\xi) \right) d\nu(\xi).$$

Writing $\nu_n = \mu_{\tau_n} * \nu$, the right side is none other than $p(n) \cdot \left\| \frac{d\nu_n}{d\nu} \right\|_2^2$, so our goal is to show that $\left\| \frac{d\nu_n}{d\nu} \right\|_2^2 \geq (1 - o_n(1))p(n)$.

With probability $(1 - o_n(1))p(n)$ the first increment of the μ_τ -random walk is drawn from $S_{n,n}$, which by Proposition 5.1 implies that $\nu(\partial\mathcal{F}_n) \geq (1 - o_n(1))p(n)$. Let $f_n = \frac{d\nu_n}{d\nu}$. Then also from Proposition 5.1, we have $\|f_n|_{\partial\mathcal{F}_n}\|_1 \geq 1 - o_n(1)$.

Applying Cauchy-Schwarz to $f_n \cdot 1_{\partial\mathcal{F}_n}$, we see that

$$1 - o_n(1) \leq \|f_n|_{\partial\mathcal{F}_n}\|_1 \leq \|f_n\|_2 \|1_{\mathcal{F}_n}\|_2 = \|f_n\|_2 (1 - o_n(1)) \sqrt{p(n)},$$

which implies $\|f_n\|_2^2 \geq (1 - o_n(1))/p(n)$.

As a result, we have

$$\sum_{g \in G} \mu_{\tau_n}(g) h_n(g) > h_n(e)$$

for n sufficiently large, so we can conclude that the μ_τ -harmonic function h_n is not μ -harmonic. Therefore the Poisson boundary of (G, μ_τ) is strictly larger than that of (G, μ) . $\square$

Remark 5.2. It is natural at this point to ask for which bounded functions $f : G \rightarrow \mathbb{R}$ there exists a randomized stopping time τ such that f is μ_τ -harmonic. One can modify our argument, letting $p(0)$ be sufficiently small and replacing Φ with a sufficiently quickly growing function, to show that with probability close to 1, the μ_τ -random walk started at any fixed g will forever lie in a subtree of the (Φ, Σ, A) -ladder forest rooted at g . As a result, one can express any bounded function on G as the pointwise limit of a sequence of bounded functions which are harmonic with respect to some randomized stopping time transform of μ .

5.1. Proof of Proposition 5.1. We start with the following lemma about record times.

Lemma 5.3. *Let p be a probability measure on $\mathbb{N}$ which has eventually simple records. Then $\mathbb{P}(n \in (R_k)_{k \geq 1}) \rightarrow 0$ as $n \rightarrow \infty$.*

Proof. If not, then there exists some $\delta > 0$ and infinitely many $n_1 < n_2 < \dots$ such that $\mathbb{P}(n_i \in (R_k)_{k \geq 1}) \geq \delta$. For each such n_i , with probability at least δ^2 , there are two consecutive record times $T_k < T_{k+1}$ such that $R_k = n_i = R_{k+1}$. Hence with positive probability infinitely many n_i are non-simple records. $\square$

Proof of Proposition 5.1. Given $\varepsilon > 0$, let $(X_i)_{i \geq 1}$ be i.i.d. samples from p and let $(g_i)_{i \geq 1}$ independent with $g_i \sim \mu_{\tau_{X_i}}$. For k_0 sufficiently large depending on ε and n sufficiently large depending on k_0 , the following holds with probability at least $1 - \varepsilon$.

- (1) n is not a record value (by Lemma 5.3)
- (2) at least one of $X_1, \dots, X_\sigma$ is equal to 0, where $\sigma = \inf\{t \geq 1, X_t \geq n\}$ (as $p(0) > 0$).
- (3) For all $k \geq k_0$, the k th record R_k is simple and $T_{k+1} \leq \Phi(R_k)$ (by Lemma 2.4).
- (4) For all $k \geq k_0$, for all $1 \leq i \leq \Phi(R_k)$, the increment g_i is drawn from $S_{X_i, k}$.
- (5) n is larger than R_{k_0} .

Let $\{w_m\}_{m \geq 1}$ be a sample path on this event. We claim that for every $m \geq 1$ and $g \in S_{n, n}$, the element gw_m is spiked and lies in $\mathcal{F}_n$. Pick k such that $R_k < n < R_{k+1}$.

Then for times $m \leq T_{k+1} - 1$, since w_m is a product of at most $\Phi(R_k) \leq \Phi(n)$ increments drawn from $\Delta_{R_k} \subset \Delta_n$, the element gw_m is spiked with spike g , hence it lies in $\mathcal{F}_n$.

For times $m \geq T_{k+1}$, choose k' minimal such that $T_{k'} \leq m \leq T_{k'+1} - 1$. As at least one of $g_1, \dots, g_{T_{k'}-1}$ is trivial, then $gw_{T_{k'}}$ is a product of at most $\Phi(R_{k'})$ increments drawn from $\Delta_{R_{k'}}$, so that gw_m is still spiked with prefix $gw_{T_{k'}-1}$ and so gw_m lies in $\mathcal{F}_n$. $\square$

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REFERENCES

- [AABV16] Gideon Amir, Omer Angel, Nicolás Matte Bon, and Bálint Virág. The Liouville property for groups acting on rooted trees. *Annales de l'Institut Henri Poincaré, Probabilités et Statistiques*, 52(4):1763 – 1783, 2016.
- [ABGK23] Gideon Amir, Guy Blachar, Maria Gerasimova, and Gady Kozma. Probabilistic laws on infinite groups, 2023.
- [Alp24a] Andrei Alpeev. Examples of measures with trivial left and non-trivial right random walk tail boundary, 2024.
- [Alp24b] Andrei Alpeev. Secret sharing on the poisson-furstenberg boundary i, 2024.
- [Alp25] Andrei Alpeev. A characterization of c^* -simplicity of countable groups via poisson boundaries, 2025.
- [Bla55] David Blackwell. On transient markov processes with a countable number of states and stationary transition probabilities. *The Annals of Mathematical Statistics*, pages 654–658, 1955.
- [BSW94] J.J.A.M. Brands, Frederik W Steutel, and Roeland Joannes Gerardus Wilms. On the number of maxima in a discrete sample. *Statistics & Probability Letters*, 20(3):209–217, 1994.
- [BV05] Laurent Bartholdi and Bálint Virág. Amenability via random walks. *Duke Mathematical Journal*, 130(1):39 – 56, 2005.
- [CFFT22] Kunal Chawla, Behrang Forghani, Joshua Frisch, and Giulio Tiozzo. The poisson boundary of hyperbolic groups without moment conditions, 2022.
- [Der75] Yves Derriennic. Marche aléatoire sur le groupe libre et frontière de martin. *Zeitschrift für Wahrscheinlichkeitstheorie und verwandte Gebiete*, 32:261–276, 1975.
- [DM56] AM Duguid and DH McLain. Fc-nilpotent and fc-soluble groups. In *Mathematical proceedings of the cambridge philosophical society*, volume 52, pages 391–398. Cambridge University Press, 1956.
- [DM61] E. B. Dynkin and M. B. Maljutov. Random walk on groups with a finite number of generators. In *Dokl. Akad. Nauk SSSR*, volume 137, pages 1042–1045, 1961.

- [Doo59] Joseph L Doob. Discrete potential theory and boundaries. *J. Math. Mech.*, 8:433–458, 1959.
- [EF25] Anna Erschler and Joshua Frisch. Poisson boundary of group extensions, 2025.
- [EFR23] Anna Erschler, Josh Frisch, and Mark Rychnovsky. Poisson boundary for upper-triangular groups, 2023.
- [Eis09] Bennett Eisenberg. The number of players tied for the record. *Statistics & probability letters*, 79(3):283–288, 2009.
- [EK23] Anna Erschler and Vadim A Kaimanovich. Arboreal structures on groups and the associated boundaries. *Geometric and Functional Analysis*, 33(3):694–748, 2023.
- [Ers04] Anna Erschler. Boundary behavior for groups of subexponential growth. *Annals of Mathematics*, pages 1183–1210, 2004.
- [Ers11] Anna Erschler. Poisson–furstenberg boundary of random walks on wreath products and free metabelian groups. *Comment. Math. Helv*, 86(1):113–143, 2011.
- [EZ20] Anna Erschler and Tianyi Zheng. Growth of periodic grigorchuk groups. *Inventiones mathematicae*, 219(3):1069–1155, 2020.
- [Fel56] William Feller. Boundaries induced by non-negative matrices. *Transactions of the American Mathematical Society*, 83(1):19–54, 1956.
- [FF18] Joshua Frisch and Pooya Vahidi Ferdowsi. Non-virtually nilpotent groups have infinite conjugacy class quotients, 2018.
- [FHTF19] Joshua Frisch, Yair Hartman, Omer Tamuz, and Pooya Vahidi Ferdowsi. Choquet-deny groups and the infinite conjugacy class property. *Annals of Mathematics*, 190(1):307–320, 2019.
- [FK24] Behrang Forghani and Vadim Kaimanovich. Random stopping times and boundary preserving transformations of random walks. unpublished manuscript, 2015–2024.
- [FMK19] Behrang Forghani and Keivan Mallahi-Karai. Positive harmonic functions of transformed random walks. *Potential Analysis*, 51:563–578, 2019.
- [For15] Behrang Forghani. *Transformed random walks*. PhD thesis, Université d’Ottawa/University of Ottawa, 2015.
- [For17] Behrang Forghani. Asymptotic entropy of transformed random walks. *Ergodic Theory and Dynamical Systems*, 37(5):1480–1491, 2017.
- [FS24] Joshua Frisch and Eduardo Silva. The poisson boundary of wreath products, 2024.
- [FT19] Behrang Forghani and Giulio Tiozzo. Random walks of infinite moment on free semigroups. *Probability Theory and Related Fields*, 175:1099–1122, 2019.
- [FTVF19] Joshua Frisch, Omer Tamuz, and Pooya Vahidi Ferdowsi. Strong amenability and the infinite conjugacy class property. *Inventiones mathematicae*, 218(3):833–851, 2019.
- [Fur63a] Harry Furstenberg. Noncommuting random products. *Transactions of the American Mathematical Society*, 108(3):377–428, 1963.
- [Fur63b] Harry Furstenberg. A poisson formula for semi-simple lie groups. *Annals of Mathematics*, 77(2):335–386, 1963.
- [Fur67] Harry Furstenberg. Poisson boundaries and envelopes of discrete groups. *Bulletin of the American Mathematical Society*, 1967.
- [Fur71] Harry Furstenberg. Random walks and discrete subgroups of lie groups. *Advances in Probability and Related Topics*, 1:1–63, 1971.
- [Fur02] Alex Furman. Random walks on groups and random transformations. In *Handbook of dynamical systems*, volume 1, pages 931–1014. Elsevier, 2002.
- [GMBT24] Elia Gorokhovsky, Nicolás Matte Bon, and Omer Tamuz. A quantitative neumann lemma for finitely generated groups. *Israel Journal of Mathematics*, 262(1):487–500, 2024.
- [HLT14] Yair Hartman, Yuri Lima, and Omer Tamuz. An abramov formula for stationary spaces of discrete groups. *Ergodic Theory and Dynamical Systems*, 34(3):837–853, 2014.
- [Hor16] Camille Horbez. The Poisson boundary of $\text{Out}(F_N)$. *Duke Mathematical Journal*, 165(2):341 – 369, 2016.
- [Jaw04] Wojciech Jaworski. Countable amenable identity excluding groups. *Canadian Mathematical Bulletin*, 47(2):215–228, 2004.
- [Kai81] Vadim A Kaimanovich. The boundaries of random walks on discrete groups. In *Theory of Probability and its Applications*, volume 26, pages 624–625. SIAM Publications, 1981.

- [Kai83a] Vadim A Kaimanovich. The differential entropy of the boundary of a random walk on a group. *Russian Mathematical Surveys*, 38(5):142–143, 1983.
- [Kai83b] Vadim A Kaimanovich. Examples of non-abelian discrete groups with non-trivial exit boundary. *Zapiski Nauchnykh Seminarov POMI*, 123:167–184, 1983.
- [Kai85] Vadim Ad Kaimanovich. An entropy criterion for maximality of the boundary of random walks on discrete groups. In *Doklady Akademii Nauk*, volume 280, pages 1051–1054. Russian Academy of Sciences, 1985.
- [Kai92] Vadim A Kaimanovich. Discretization of bounded harmonic functions on riemannian manifolds and entropy. In *Proceedings of the International Conference on Potential Theory*, pages 212–223. De Gruyter, Berlin, 1992.
- [Kai96] Vadim Kaimanovich. Boundaries of invariant markov operators: the identification problem. *London Mathematical Society Lecture Note Series*, pages 127–176, 1996.
- [Kai00] Vadim A Kaimanovich. The poisson formula for groups with hyperbolic properties. *Annals of Mathematics*, pages 659–692, 2000.
- [Kai24] Vadim Kaimanovich. Liouville property and poisson boundary of random walks with infinite entropy: What’s amiss? *Functional Analysis and Its Applications*, 58(2):160–181, 2024.
- [KV83] Vadim A Kaimanovich and Anatoly M Vershik. Random walks on discrete groups: boundary and entropy. *The Annals of Probability*, 11(3):457–490, 1983.
- [LBMB18] Adrien Le Boudec and Nicolás Matte Bon. Subgroup dynamics and c^* -simplicity of groups of homeomorphisms. In *Annales scientifiques de l’École Normale Supérieure*, volume 51, pages 557–602, 2018.
- [Led85] François Ledrappier. Poisson boundaries of discrete groups of matrices. *Israel journal of mathematics*, 50:319–336, 1985.
- [LP21] Russell Lyons and Yuval Peres. Poisson boundaries of lamplighter groups: proof of the kaimanovich–vershik conjecture. *J. Eur. Math. Soc. (JEMS)*, 23(4):1133–1160, 2021.
- [LS84] Terry Lyons and Dennis Sullivan. Function theory, random paths and covering spaces. *Journal of differential geometry*, 19(2):299–323, 1984.
- [McL56] DH McLain. Remarks on the upper central series of a group. *Glasgow Mathematical Journal*, 3(1):38–44, 1956.
- [MT18] Joseph Maher and Giulio Tiozzo. Random walks on weakly hyperbolic groups. *Journal für die reine und angewandte Mathematik (Crelles Journal)*, 2018(742):187–239, 2018.
- [Qi97] Yongcheng Qi. A note on the number of maxima in a discrete sample. *Statistics & probability letters*, 33(4):373–377, 1997.
- [Roh67] VA Rohlin. Lectures on the entropy theory of transformations with invariant measure. *Uspehi Mat. Nauk*, 22(5 (137)):3–56, 1967.
- [Sil24] Eduardo Silva. The poisson boundary of lampshuffler groups. *Mathematische Zeitschrift*, 308(3):46, 2024.
- [Ver73] Wim Vervaat. Limit theorems for records from discrete distributions. *Stochastic Processes and their Applications*, 1(4):317–334, 1973.
- [VKS25] Martín Gilabert Vio, Cosmas Kravaris, and Eduardo Silva. The poisson boundary of thompson’s group t is not the circle, 2025.
- [Zhe22] Tianyi Zheng. Asymptotic behaviors of random walks on countable groups. In *Proc. Int. Cong. Math*, volume 4, pages 3340–3365, 2022.

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